
The Mathematics of Hiview and Equity

Introduction

The two computer programs Hiview and Equity are both based on multi-criteria decision analysis, the subject of Keeney and Raiffa's 1976 classic book. In that book, the authors extended the axioms of decision theory, which lead to the expected utility model, to provide for consequences characterised by multiple criteria. They pointed out, and Keeney (1992) subsequently elaborated, that decisions are made to realise objectives, but that objectives often conflict. How to deal with that conflict is the subject of multi-criteria decision analysis. Their approach is particularly attractive because it accommodates consequences that are both uncertain and appraised differently depending on the criteria considered.

Some theory

The basic theory results in different practical procedures depending on whether certain assumptions are met. Thus, it is possible to consider either utility, which is an expression of subjective value along with the decision maker's risk attitude, or value alone, where value is defined as the extent to which the consequence of a decision achieves a given objective. The utility/value distinction is frequently not observed in practice, with most decision analysts focussing on value alone.

The theory can also be used in cases where values are interdependent; the value assigned to a consequence on one criterion depends on the value assigned to another criterion. For example, for many people the relative values of different wines depend on the values of the main dishes. Such interdependence of values requires multiplicative terms in the models, whereas independence of values leads to simple additive models. It is important to note that value independence is often observed even when the consequences are correlated in the real world. For example, the poshness of a car's interior and the cost of the car are negatively correlated across a range of cars, yet for many people they are value independent: more poshness is preferred, all other things considered equal, and less cost is preferred, all other things considered equal. But posh cars are usually more expensive. Thus, value independence is a weaker condition than statistical independence.

Value independence is so frequently observed in real-world problems, that the simple additive model is usually found in published case studies. In any event, von Winterfeldt and Edwards (1986) argue that even when value independence is violated, the results of the simpler additive multi-criteria model are still valid. Finally, it is usually simple to re-structure the criteria, combining or re-expressing those that are not value independent, to create a set of value-independent criteria. For these reasons, the designers of Hiview and Equity elected to use only the simple additive mathematics. It then behoves the user to ensure that the criteria are reasonably value independent.

The additive model

It's simple. Scores are multiplied by weights and the products summed. Let v_{ij} represent the value associated with the value of option i on criterion j . Let w_j represent the weight assigned to criterion j . Then, the overall value of option i is simply:

$$V_i = \sum_j w_j v_{ij} \quad (1)$$

In words, the scores assigned to the consequence of an option on all the criteria are multiplied by the respective weights assigned to the criteria and those products summed across all the criteria. How this is implemented differs in Hiview and Equity.

Implementation in Hiview

Hiview allows a hierarchical representation of objectives and criteria. Criteria are clustered under 'parent' nodes. All options are scored on all the criteria under the parent. The criteria are weighted. Whatever weighting system is used, Hiview assumes that the ratios of those weights are sacrosanct, and it normalises them by dividing the weight on each criterion by the sum of the weights on all the criteria under that node. This preserves the ratios, and yields a set of weights that sum to 1. Those weights are used in the above equation to give a single, weighted average scale for the parent node. A Hiview matrix from the 'Shampoo' model shows how equation (1) is implemented. (Note that the weights are shown here normalised. They were assessed as 60 and 120, respectively for Size and Share.)

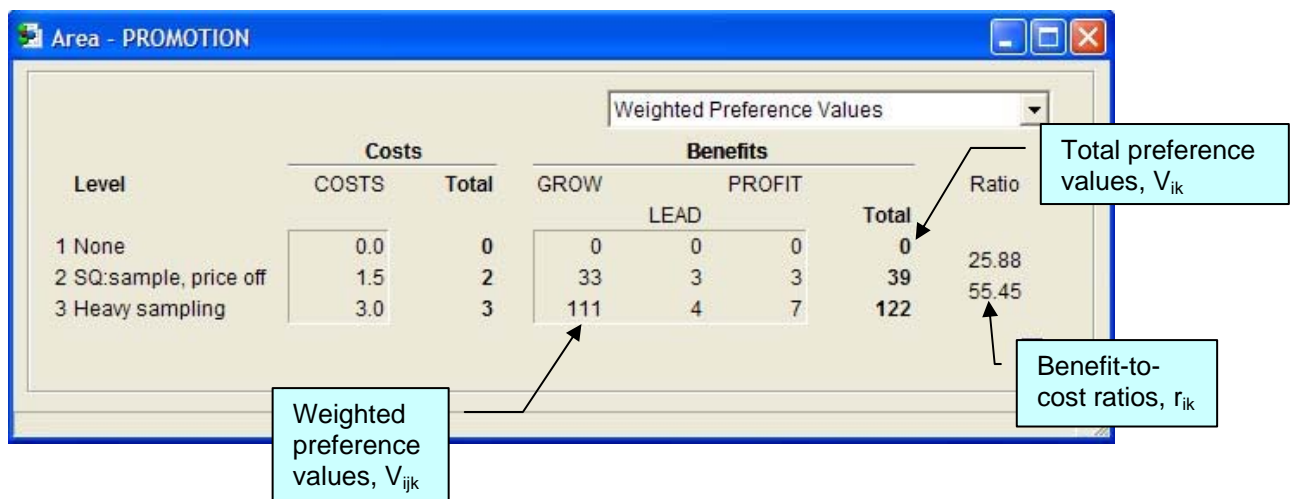
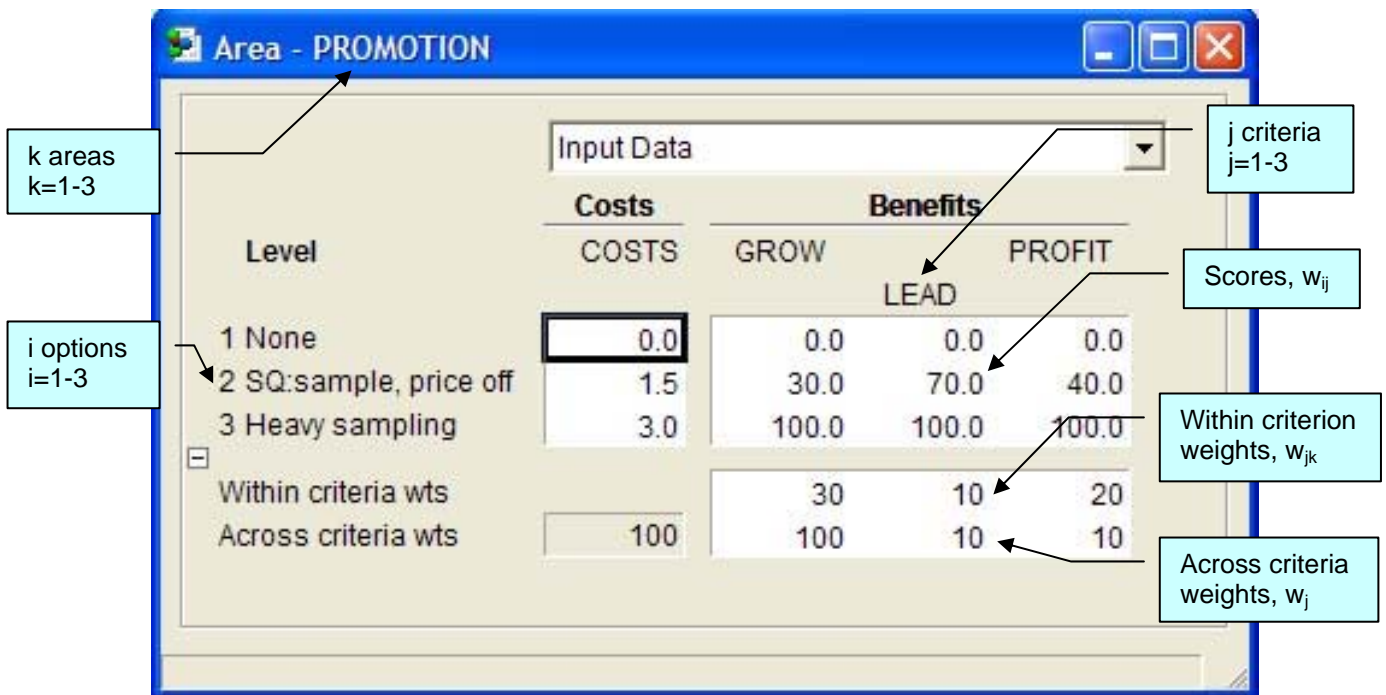
The screenshot shows a window titled "GROW MARKET Node Data" with a dropdown menu set to "GROW MARKET Data Breakdown". The table below is annotated with callouts: "j criteria j=1,2" points to the first column; "Criterion weights, w_j" points to the "Weight" column; "i options i=1-6" points to the "Distribution Hi qual" column; "Scores, w_{ij}" points to the numerical values in the table; and "Overall values, V_i" points to the "Cumulative Weight" column.

GROW MARKET	Weight	Consumers		Products		Distribution		Cumulative Weight
		Status quo	Promotion	Status quo	Promotion	Status quo	Hi qual	
Size*	33	10	100	10	40	40	0	4.6
Share*	67	50	30	50	70	0	100	9.4
TOTAL	100	37	53	37	60	13	67	14.1

Weights at the parent nodes can be assessed separately, or Hiview can be instructed to use the sum of the lower-level weights. The latter approach is applied across all parent nodes whose children criteria are weighted by comparing them to a common standard, usually the one criterion associated with the largest swing in value from least to most preferred options. Typically, this method is used, for example, within the benefit cluster of criteria, but separate weights are then assessed as between the Benefit node, and, say, a Cost node whose criteria were assessed relative to a single Cost criterion. It is important for the user to understand that Hiview does not know what methods of scoring and weighting are used. Thus, care must be exercised to ensure that the simple calculation above does not give misleading results.

Implementation in Equity

Equity is more complex because a double weighting system is used, one expressing the relative weights associated with the Benefit criteria, as in Hiview, and another giving relative weights on a given criterion but across the areas. The weights associated with the Benefit criteria are called 'across criteria weights', while the weights from one area to the next on a given criterion are called the 'within criterion weights'. Again, the simple additive model is assumed, but equation (1) is expanded to include the within criterion weight, w_{jk} , the weight on criterion j for area k . The input score for option i on criterion j in area k is represented by v_{ijk} (not shown below). Shampoo again:



Equity converts input scores to weighted preference values V_{ijk} by applying this additive model:

$$V_{ijk} = 10 \frac{w_j w_{jk} v_{ijk}}{\sum_j \sum_k w_j w_{jk}} \quad (2)$$

First consider the denominator. For each criterion, Equity multiplies the across criterion weight w_j by the within-criterion weight w_{jk} . It sums these products for all criteria in all areas; that is the double sum in the denominator of equation (2). For the numerator, it multiplies each input score, v_{ijk} , by the product of the across and within criterion weights appropriate for that score. It then multiplies this result by 10 so the weighted preference values are shown out of 1000 rather than the input

100. However, unlike Hiview, in which the weights are always normalised at each node, Equity works differently. The program first examines the assessed (input) within criterion weights across all the areas for a given criterion and sets the largest one to 100, scaling all the others proportionately. In other words, it preserves the ratios of the weights from one area to the next and ensures that the largest within criterion weight, for a given criterion, is set to 100. It repeats this process for the remaining benefit criteria. It does the same thing for the assessed (input) across criteria weights, setting the largest to 100. Equity *then* performs the calculation shown in equation (2). This set of calculations must be understood by the user to ensure that the weights are assessed in a way that is consistent with the mathematics. Then, any approach can be used for the within criterion weights, provided that the ratios of the weights are meaningful.

Next, Equity calculates the total preference value, V_{ik} , for each level in each area.

$$V_{ik} = \sum_j V_{ijk} \quad (3)$$

This is just the sum across the criteria of the values V_{ijk} for a given level in a given area.

Finally, Equity calculates the benefit-to-cost ratios by dividing the difference in the V_{ik} values from one level to the next by the cost difference.

$$r_{ik} = \frac{V_{ik} - V_{(i-1)k}}{C_{ik} - C_{(i-1)k}} \quad \text{for } i > 1 \quad (4)$$

Those are the basic calculations. Of course, each program does more, but given the above, the nature of the additional calculations should be clear.

References

- Keeney, R. L. (1992). *Value-Focused Thinking: A Path to Creative Decisionmaking*. Cambridge, MA: Harvard University Press.
- Keeney, R. L., & Raiffa, H. (1976). *Decisions With Multiple Objectives: Preferences and Value Tradeoffs*. New York: John Wiley.
- von Winterfeldt, D., & Edwards, W. (1986). *Decision analysis and behavioral research*. Cambridge: Cambridge University Press.